THE TWO DIMENSIONAL HANNAY-BERRY MODEL

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Abstract

The main goal of this paper is to construct the Hannay-Berry model of quantum mechanics, on a two dimensional symplectic torus. We construct a simultaneous quantization of the algebra of functions and the linear symplectic group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. We obtain the quantization via an action of Γ on the set of equivalence classes of irreducible representations of Rieffel's quantum torus $\mathcal{A}_\hbar$. For $\hbar \in \mathbb{Q}$ this action has a unique fixed point. This gives a canonical projective equivariant quantization. There exists a Hilbert space on which both Γ and $\mathcal{A}_\hbar$ act equivariantly. Combined with the fact that every projective representation of Γ can be lifted to a linear representation, we also obtain linear equivariant quantization.

0 Introduction

0.1 Motivation

In the paper "*Quantization of linear maps on the torus - Fresnel diffraction by a periodic grating*", published in 1980 (cf. [HB]), the physicists J. Hannay and M.V. Berry explore a model for quantum mechanics on the 2-dimensional torus. Hannay and Berry suggested to quantize simultaneously the functions on the torus and the linear symplectic group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. They found (cf. [HB],[Me]) that the theta subgroup $\Gamma_\Theta \subset \Gamma$ is the largest that one can quantize and asked (cf. [HB],[Me]) whether the quantization of Γ satisfy a multiplicativity property (i.e., is a linear representation of the group). In this paper we want to *construct* the Hannay-Berry's model for the bigger group of symmetries, i.e., the whole symplectic group Γ. The central *question* is whether there exists a Hilbert space on which a deformation of the algebra of functions and the linear symplectic group Γ both act in a compatible way.

0.2 Results

In this paper we give an *affirmative* answer to the existence of the quantization procedure. We show a construction (Theorem 0.3, Corollary 0.4 and Theorem 0.5) of the canonical equivariant quantization procedure for rational Planck constants. It is *unique* as a projective quantization (see definitions below). We show that the projective representation of Γ can be lifted in exactly 12 different ways to a linear representation (to obey the multiplicativity property). These are the first examples of such equivariant quantization for the whole symplectic group Γ. Our construction slightly *improves* the known constructions [HB, Me, KR1] for which the group of quantizable elements is $\Gamma_\Theta \subset \Gamma$ and gives a *positive* answer to the Hannay-Berry question on the linearization of the projective representation of the group of quantizable elements. (cf. [HB], [Me]). Previously it was shown by Mezzadri and Kurlberg-Rudnick (cf. [Me], [KR1]) that one can construct an equivariant quantization for the theta subgroup, in case when the Planck constant is of the form $\hbar = \frac{1}{N}, \ N \in \mathbb{N}$.

0.2.1 Classical torus

Let $(\mathbf{T}, \omega)$ be the two dimensional symplectic torus. Together with its linear symplectomorphisms $\Gamma \backsimeq \mathrm{SL}_2(\mathbb{Z})$ it serves as a simple model of classical mechanics (a compact version of the phase space of the harmonic oscillator). More precisely, let $\mathbf{T} = \mathrm{W}/\Lambda$ where W is a two dimensional real vector space, i.e., $\mathrm{W} \simeq \mathbb{R}^2$ and Λ is a rank two lattice in W, i.e., $\Lambda \simeq \mathbb{Z}^2$. We obtain the symplectic form on $\mathbf{T}$ by taking a non-degenerate symplectic form on W:

$$\omega : \mathrm{W} \times \mathrm{W} \longrightarrow \mathbb{R}.$$

We require ω to be integral, namely $\omega : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ and normalized, i.e., $\mathrm{Vol}(\mathbf{T}) = 1$.

Let $\mathrm{Sp}(\mathrm{W}, \omega)$ be the group of linear symplectomorphisms, i.e., $\mathrm{Sp}(\mathrm{W}, \omega) \simeq \mathrm{SL}_2(\mathbb{R})$. Consider the subgroup $\Gamma \subset \mathrm{Sp}(\mathrm{W}, \omega)$ of elements that preserve the lattice Λ, i.e., $\Gamma(\Lambda) \subseteq \Lambda$. Then $\Gamma \simeq \mathrm{SL}_2(\mathbb{Z})$. The subgroup Γ is the group of linear symplectomorphisms of $\mathbf{T}$.
We denote by $\Lambda^* \subseteq \mathrm{W}^*$ the dual lattice:

$$\Lambda^* := \{\xi \in \mathrm{W}^* | \ \xi(\Lambda) \subset \mathbb{Z}\}.$$

The lattice Λ^* is identified with the lattice $\mathbf{T}^\vee := \mathrm{Hom}(\mathbf{T}, \mathbb{C}^*)$ of characters of $\mathbf{T}$ by the following map:

$$\xi \in \Lambda^* \longmapsto e^{2\pi i <\xi, \cdot>} \in \ \mathbf{T}^\vee.$$

The form ω allows us to identify the vector spaces W and W^*. For simplicity we will denote the induced form on W^* also by ω.

0.2.2 Equivariant quantization of the torus

We will construct a particular type of quantization procedure for the functions. Moreover this quantization will be equivariant with respect to the action of the "classical symmetries" Γ:

Definition 0.1 *By Weyl quantization of $\mathcal{A}$ we mean a family of $\mathbb{C}$-linear, $*-$ morphisms $\pi_{\hbar} : \mathcal{A} \longrightarrow End(\mathcal{H}_{\hbar})$, $\hbar \in \mathbb{R}$, where $\mathcal{H}_{\hbar}$ is a Hilbert space, s.t. the following property holds:*

$$\pi_{\hbar}(\xi + \eta) = e^{\pi i \hbar w(\xi,\eta)} \pi_{\hbar}(\xi) \pi_{\hbar}(\eta)$$

for all $\xi, \eta \in \Lambda^*$ and $\hbar \in \mathbb{R}$.

This type of quantization procedure will obey the "usual" properties (cf. [D4]):

$$\begin{aligned} ||\pi_{\hbar}(fg) - \pi_{\hbar}(f)\pi_{\hbar}(g)||_{\mathcal{H}_{\hbar}} &\longrightarrow 0, \quad as\ \hbar \to 0, \\ ||\frac{i}{\hbar}[\pi_{\hbar}(f), \pi_{\hbar}(g)] - \pi_{\hbar}(\{f,g\})||_{\mathcal{H}_{\hbar}} &\longrightarrow 0, \quad as\ \hbar \to 0. \end{aligned}$$

where $\{,\}$ is the Poisson brackets on functions.

Definition 0.2 *By equivariant quantization of $\mathbf{T}$ we mean a quantization of $\mathcal{A}$ with additional maps $\rho_{\hbar} : \Gamma \longrightarrow \mathrm{U}(\mathcal{H}_{\hbar})$ s.t. the following equivariant property (called Egorov's identity) holds:*

$$\rho_{\hbar}(B)^{-1} \pi_{\hbar}(f) \rho_{\hbar}(B) = \pi_{\hbar}(f \circ B) \tag{0.2.1}$$

for all $\hbar \in \mathbb{R}$, $f \in \mathcal{A}$ and $B \in \Gamma$. Here $\mathrm{U}(\mathcal{H}_{\hbar})$ is the group of unitary operators on $\mathcal{H}_{\hbar}$. If $(\rho_{\hbar}, \mathcal{H}_{\hbar})$ is a projective (respectively linear) representation of the group Γ then we call the quantization projective (respectively linear).

The idea of the construction is as follows: We use a "deformation" of the algebra $\mathcal{A}$ of functions on $\mathbf{T}$. We define an algebra $\mathcal{A}_{\hbar}$, usually called the two dimensional non-commutative torus (cf. [Ri]). If $\hbar = \frac{M}{N} \in \mathbb{Q}$, then we will see that all irreducible representations of $\mathcal{A}_{\hbar}$ have dimension N. We denote by $\mathrm{Irr}(\mathcal{A}_{\hbar})$ the set of equivalence classes of irreducible algebraic representations of the quantized algebra. We will see that $\mathrm{Irr}(\mathcal{A}_{\hbar})$ is a set "equivalent" to a torus.

The group Γ naturally acts on a quantized algebra $\mathcal{A}_{\hbar}$ and hence on the set $\mathrm{Irr}(\mathcal{A}_{\hbar})$. Let $\hbar = \frac{M}{N}$ with $\gcd(M,N) = 1$. The following holds:

Theorem 0.3 (Canonical equivariant representation) *There exists a unique (up to isomorphism) N-dimensional irreducible representation $(\pi_{\hbar}, \mathcal{H}_{\hbar})$ of $\mathcal{A}_{\hbar}$ for which its equivalence class is fixed by Γ.*

This means that:

$$\pi_{\hbar} \backsimeq \pi_{\hbar}^{B}$$

for all $B \in \Gamma$.

Since the canonical representation $(\pi_{\hbar}, \mathcal{H}_{\hbar})$ is irreducible, by Schur's lemma we get the canonical projective representation of Γ compatible with $\pi_{\hbar}$:

Corollary 0.4 (Canonical projective representation) *There exists a unique projective representation* $\rho_{\mathrm{p}} : \Gamma \longrightarrow \mathrm{PGL}(\mathcal{H}_{\hbar})$ *s.t.:*

$$\rho_{\mathrm{p}}(B)^{-1}\pi_{\hbar}(f)\rho_{\mathrm{p}}(B) = \pi_{\hbar}(f \circ B)$$

for all $f \in \mathcal{A}$ *and* $B \in \Gamma$.

Remark. Corollary 0.4 is an improvement to the known constructions (cf. [HB, Me, KR1]) which has the group $\Gamma_{\Theta} := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid ab = cd = 0 \ (2)\}$ as the group of quantizable elements.

Using a result of Coxeter-Moser [CM] about the structure of the group Γ we get:

Theorem 0.5 (Linearization) *The projective representation* ρ_{p} *can be lifted to a linear representation in exactly* 12 *different ways.*

Remark. The existence of the linear representation $\rho_{\hbar}$ in Theorem 0.5 answers Hannay-Berry's question (cf. [HB, Me]) on the multiplicativity of the map $\rho_{\hbar}$.

Summary. For $\hbar \in \mathbb{Q}$ let $(\rho_{\hbar}, \pi_{\hbar}, \mathcal{H}_{\hbar})$ be the canonical (projective) equivariant quantization of $\mathbf{T}$. We can endow the space $\mathcal{H}_{\hbar}$ with a canonical unitary structure s.t. $\pi_{\hbar}$ is a $*$-representation and $\rho_{\hbar}$ is unitary. This "family" of $*-$representations of $\mathcal{A}_{\hbar}$ is by definition a Weyl quantization of the functions on the torus. The above results show the existence of a canonical projective equivariant quantization of the torus, and the existence of a linear equivariant quantization of the torus.

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1 Construction

We consider the algebra $\mathcal{A} := C^\infty(\mathbf{T})$ of smooth complex valued function on the torus and the dual lattice $\Lambda^* := \{\xi \in V^* |\ \xi(\Lambda) \subset \mathbb{Z}\}$. Let $<,>$ be the pairing between W and W*. The map $\xi \mapsto s(\xi)$ where $s(\xi)(x) := e^{2\pi i <x,\xi>},\ x \in \mathbf{T}$ and $\xi \in \Lambda^*$ defines a canonical isomorphism between Λ^* and the group $\mathbf{T}^\vee := \mathrm{Hom}(\mathbf{T}, \mathbb{C}^*)$ of characters of $\mathbf{T}$.

1.1 The quantum tori

Fix $\hbar \in \mathbb{R}$. The Rieffel's quantum torus (cf. [Ri]) is the non-commutative algebra $\mathcal{A}_\hbar$ defined over $\mathbb{C}$ by generators $\{s(\xi),\ \xi \in \Lambda^*\}$, and relations:

$$s(\xi + \eta) = e^{\pi i \hbar \omega(\xi,\eta)} s(\xi) s(\eta)$$

for all $\xi, \eta \in \Lambda^*$.

Note that the lattice Λ^* serves, using the map $\xi \mapsto s(\xi)$, as a basis for the algebra $\mathcal{A}_\hbar$. This induces an identification of vector spaces $\mathcal{A}_\hbar \backsimeq \mathcal{A}$ for every $\hbar$. We will use this identification in order to view elements of the (commutative) space $\mathcal{A}$ as members of the (non-commutative) space $\mathcal{A}_\hbar$.

1.2 Weyl quantization

To get a Weyl quantization of $\mathcal{A}$ we use a specific one-parameter family of representations (see subsection 1.4 below) of the quantum tori. This defines an operator $\pi_\hbar(\xi)$ for every $\xi \in \Lambda^*$. We extend the construction to every function $f \in \mathcal{A}$ using the Fourier theory. Suppose:

$$f = \sum_{\xi \in \Lambda^*} a_\xi \cdot \xi$$

is its Fourier expansion. Then we define its *Weyl quantization* by:

$$\pi_\hbar(f) := \sum_{\xi \in \Lambda^*} a_\xi \pi_\hbar(\xi).$$

The convergence of the last series is due to the rapid decay of the Fourier coefficients of the function f.

1.3 Projective equivariant quantization

The group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ acts on Λ preserving ω. Hence Γ acts on $\mathcal{A}_\hbar$ and the formula of this action is $s^B(\xi) := s(B\xi)$. Given a representation $(\pi_\hbar, \mathcal{H}_\hbar)$ of $\mathcal{A}_\hbar$

and an element $B \in \Gamma$, define $\pi_{\hbar}^{B}(s(\xi)) := \pi_{\hbar}(s^{B^{-1}}(\xi))$. This formula induces an action of Γ on the set $\mathrm{Irr}(\mathcal{A}_{\hbar})$ of equivalence classes of irreducible algebraic representations of $\mathcal{A}_{\hbar}$.

Lemma 1.1 *All irreducible representations of $\mathcal{A}_{\hbar}$ are N-dimensional.*

Now, suppose $(\pi_{\hbar}, \mathcal{A}_{\hbar}, \mathcal{H}_{\hbar})$ is an irreducible representation for which its equivalence class is fixed by the action of Γ. This means that for any $B \in \Gamma$ we have $\pi_{\hbar} \backsimeq \pi_{\hbar}^{B}$, so by definition there exists an operator $\rho_{\hbar}(B) \in \mathrm{GL}(\mathcal{H}_{\hbar})$ such that:

$$\rho_{\hbar}(B)^{-1}\pi_{\hbar}(\xi)\rho_{\hbar}(B) = \pi_{\hbar}(B\xi)$$

for all $\xi \in \Lambda^*$. This implies the Egorov identity (0.2.1) for any function . Now, since $(\pi_{\hbar}, \mathcal{H}_{\hbar})$ is an irreducible representation then by Schur's lemma for every $B \in \Gamma$ the operator $\rho_{\hbar}(B)$ is uniquely defined up to a scalar. This implies that $(\rho_{\hbar}, \mathcal{H}_{\hbar})$ is a projective representation of Γ.

1.4 The canonical equivariant quantization

In what follows we consider only the case $\hbar \in \mathbb{Q}$. We write $\hbar$ in the form $\hbar = \frac{M}{N}$ with $\gcd(M, N) = 1$.

Proposition 1.2 *There exists a unique $\pi_{\hbar} \in \mathrm{Irr}(\mathcal{A}_{\hbar})$ which is a fixed point for the action of Γ.*

1.5 Unitary structure

Note that $\mathcal{A}_{\hbar}$ becomes a $*-$ algebra using the formula $s(\xi)^* := s(-\xi)$. Let $(\pi_{\hbar}, \mathcal{H}_{\hbar})$ be the canonical representation of $\mathcal{A}_{\hbar}$.

Remark 1.3 *There exists a canonical (unique up to scalar) unitary structure on $\mathcal{H}_{\hbar}$ for which $\pi_{\hbar}$ is a $*-$representation.*

1.6 Realization

Choosing a symplectic basis for Λ^* we get the identifications $\Lambda^* \backsimeq \mathbb{Z} \oplus \mathbb{Z}$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. We will consider the realization on the Hilbert space:

$$\mathcal{H} := \mathrm{L}^2(\mathbb{Z}/N\mathbb{Z}).$$

1.6.1 Formula for π

The representation π is given by:

$$[\pi(\mathrm{m},\mathrm{n})\mathrm{f}]\,(\mathrm{x}) = \alpha(\mathrm{m},\mathrm{n})\psi(\mathrm{nx})\mathrm{f}(\mathrm{x}+\mathrm{m}),$$

where $\alpha(\mathrm{m},\mathrm{n}) := (-1)^{M(\mathrm{m}+\mathrm{n})}e^{\pi i\hbar \mathrm{mn}}$ and $\psi(t)$ denote the additive character $\psi(t) := e^{2\pi i\hbar t}$ on $\mathbb{Z}/N\mathbb{Z}$.

1.6.2 formula for ρ

The projective representation ρ is described by the following formulas:

$$\left[\rho\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\mathrm{f}\right](\mathrm{x}) \quad = \quad \mathtt{Q}(\mathrm{x})\mathrm{f}(\mathrm{x}),$$

where $\mathtt{Q}(\mathrm{x}) := (-1)^{\varepsilon \mathrm{x}}e^{\pi i\hbar \mathrm{x}^2}$, with $\varepsilon := MN \pmod 2$, and:

$$\left[\rho\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathrm{f}\right](\mathrm{x}) \quad = \quad \widehat{\mathrm{f}}(\mathrm{x}),$$

where $\widehat{\mathrm{f}}$ denote the Fourier transform:

$$\widehat{\mathrm{f}}(\mathrm{x}) := \frac{1}{\sqrt{N}}\sum_{\mathrm{y}\in\mathbb{Z}/NZ}\mathrm{f}(\mathrm{y})\psi(\mathrm{yx}).$$

2 Proofs

2.1 Proof of Lemma 1.1

Suppose $(\pi_{\hbar}, \mathcal{H}_\hbar)$ is an irreducible representation of $\mathcal{A}_\hbar$.

Step 1. First we show that $\mathcal{H}_\hbar$ is finite dimensional. $\mathcal{A}_\hbar$ is a finite module over $Z(\mathcal{A}_\hbar) = \{s(N\xi),\ \xi \in \Lambda^*\}$ which is contained in the center of $\mathcal{A}_\hbar$. Because $\mathcal{H}_\hbar$ has at most countable dimension (as a quotient space of $\mathcal{A}_\hbar$) and $\mathbb{C}$ is uncountable then by Kaplansky's trick (cf. [MR]) $Z(\mathcal{A}_\hbar)$ acts on $\mathcal{H}_\hbar$ by scalars. Hence dim $\mathcal{H}_\hbar < \infty$.

Step 2. We show that $\mathcal{H}_\hbar$ is N-dimensional. Choose a basis (e_1, e_2) of Λ^* s.t. $\omega(e_1, e_2) = 1$. Suppose $\lambda \neq 0$ is an eigenvalue of $\pi_{\hbar}(e_1)$ and denote by $\mathcal{H}_\lambda$ the corresponding eigenspace. We have the following commutation relation

$\pi_\hbar(e_1)\pi_\hbar(e_2) = \gamma\pi_\hbar(e_2)\pi_\hbar(e_1)$ where $\gamma := e^{-2\pi i \frac{M}{N}}$. Hence $\pi_\hbar(e_2) : \mathcal{H}_{\gamma^j\lambda} \longrightarrow \mathcal{H}_{\gamma^{j+1}\lambda}$, and because $\gcd(M,N) = 1$ then $\mathcal{H}_{\gamma^i\lambda} \neq \mathcal{H}_{\gamma^j\lambda}$ for $0 \leq i \neq j \leq N-1$. Now, let $v \in \mathcal{H}_\lambda$ and recall that $\pi_\hbar(e_2)^N = \pi_\hbar(Ne_2)$ is a scalar operator. Then the space $\mathrm{span}\{v, \pi_\hbar(e_2)v, \ldots, \pi_\hbar(e_2)^{N-1}v\}$ is N-dimensional $\mathcal{A}_\hbar-$invariant subspace hence it equals $\mathcal{H}_\hbar$. ■

2.2 Proof of Proposition 1.2

Let us show the existence of a unique fixed point for the action of Γ on $\mathrm{Irr}(\mathcal{A}_\hbar)$.

Suppose $(\pi_\hbar, \mathcal{H}_\hbar)$ is an irreducible representation of $\mathcal{A}_\hbar$. By Schur's lemma for every $\xi \in \Lambda^*$ the operator $\pi_\hbar(N\xi)$ is a scalar operator, i.e., $\pi_\hbar(N\xi) = q_{\pi_\hbar}(\xi) \cdot \mathrm{I}$. We have $\pi_\hbar(0) = \mathrm{I}$ and hence $q_{\pi_\hbar}(\xi) \neq 0$ for all $\xi \in \Lambda^*$. Thus to any irreducible representation we have attached a scalar function $q_{\pi_\hbar} : \Lambda^* \longrightarrow \mathbb{C}^*$. Consider the set $Q_\hbar$ of *twisted characters* of Λ^*:

$$Q_\hbar := \{q : \Lambda^* \longrightarrow \mathbb{C}^*,\ q(\xi + \eta) = (-1)^{MNw(\xi,\eta)} q(\xi) q(\eta)\}.$$

The group Γ acts naturally on this space by $q^B(\xi) := q(B^{-1}\xi)$. It is easy to see that we have defined a map $\mathbf{q} : \mathrm{Irr}(\mathcal{A}_\hbar) \longrightarrow Q_\hbar$ given by $\pi_\hbar \mapsto q_{\pi_\hbar}$ and it is obvious that this map is compatible with the action of Γ. We use the space of twisted characters in order to give a description for the set $\mathrm{Irr}(\mathcal{A}_\hbar)$:

Lemma 2.1 *The map $\pi_\hbar \mapsto q_{\pi_\hbar}$ is a Γ-equivariant bijection:*

$$\mathbf{q} : \mathrm{Irr}(\mathcal{A}_\hbar) \longrightarrow Q_\hbar.$$

Now, Proposition 1.2 follows from the following claim:

Claim 2.2 *There exists a unique $q_o \in Q_\hbar$ which is a fixed point for the action of Γ.*

Proof of Lemma 2.1. Step 1. The map $\mathbf{q}$ is surjective. Denote by $\mathbb{T} := \mathrm{Hom}(\Lambda^*, \mathbb{C}^*)$ the group of complex characters of Λ^*. We define an action of $\mathbb{T}$ on $\mathrm{Irr}(\mathcal{A}_\hbar)$ and on $Q_\hbar$ by $\pi_\hbar \mapsto \chi\pi_\hbar$ and $q \mapsto \chi^N q$, where $\chi \in \mathbb{T}$, $\pi_\hbar \in \mathrm{Irr}(\mathcal{A}_\hbar)$ and $q \in Q_\hbar$. The map $\mathbf{q}$ is clearly a $\mathbb{T}$-equivariant map with respect to these actions. Since $\mathbf{q}$ is $\mathbb{T}$-equivariant, it is enough to show that the action of $\mathbb{T}$ on $Q_\hbar$ is transitive. Suppose $q_1, q_2 \in Q_\hbar$. By definition there exists a character $\chi_1 \in \mathbb{T}$ for which $\chi_1 q_1 = q_2$. Let χ be one of the N's roots of χ_1 then $\chi^N q_1 = q_2$.

Step 2. The map $\mathbf{q}$ is one to one. Suppose $(\pi_\hbar, \mathcal{H}_\hbar)$ is an irreducible representation of $\mathcal{A}_\hbar$. It is easy to deduce from the proof of Lemma 1.1 (Step 2)

that for $\xi \notin N\Lambda^*$ we have $\mathrm{tr}(\pi_\hbar(\xi)) = 0$. But we know from character theory that an isomorphism class of a finite dimensional irreducible representation of an algebra is recovered from its character. This completes the proof of Lemma 2.1. ■

Proof of Claim 2.2. *Uniqueness.* Fix $q \in Q_\hbar$. The map $\chi \mapsto \chi q$ give a bijection of $\mathbb{T}$ with $Q_\hbar$. But the trivial character $\mathbf{1} \in \mathbb{T}$ is the unique fixed point for the action of Γ on $\mathbb{T}$.
Existence. Choose a basis (e_1, e_2) of Λ^* s.t. $\omega(e_1, e_2) = 1$. This allows to identify Λ^* with $\mathbb{Z} \oplus \mathbb{Z}$. It is easy to see that the function:

$$q_o(m, n) = (-1)^{MN(mn+m+n)}$$

is a twisted character which is fixed by Γ. This completes the proof of Claim 2.2 and of Proposition 1.2. ■

2.3 Proof of Theorem 0.5

The theorem follows from the following proposition:

Proposition 2.3 *Fix a projective representation* $\rho_{\mathrm{p}} : \Gamma \longrightarrow \mathrm{GL}(\mathcal{H}_\hbar)$. *Then it can be lifted to a linear representation in exactly* 12 *ways.*

Proof. Existence. We want to find constants $c(B)$ for every $B \in \Gamma$ s.t. $\rho_\hbar := c(\cdot)\rho_{\mathrm{p}}$ is a linear representation of Γ. This is possible to carry out due to the following fact:

Lemma 2.4 ([CM]) *The group* Γ *is isomorphic to the group generated by three letters* S, B *and* Z *subjected to the relations:* $Z^2 = 1$ *and* $S^2 = B^3 = Z$.

Lemma 2.4 $\Rightarrow$ Existence. We need to find constants c_Z, c_B, c_S so that the operators $\rho_\hbar(Z) := c_Z\rho_{\mathrm{p}}(Z)$, $\rho_\hbar(B) := c_B\rho_{\mathrm{p}}(B)$, $\rho_\hbar(S) := c_S\rho_{\mathrm{p}}(S)$ will satisfy the identities:

$$\rho_\hbar(Z)^2 = I,\ \rho_\hbar(B)^3 = \rho_\hbar(Z),\ \rho_\hbar(S)^2 = \rho_\hbar(Z).$$

This can be done by taking appropriate scalars.

Now, fix one lifting ρ_0. Then for the collection of operators $\rho_\hbar(B)$ which lifts ρ_{p} define a function $\chi(B)$ by $\rho_\hbar(B) = \chi(B)\rho_0(B)$. It is obvious that $\rho_\hbar$ is a representation if and only if χ is a character. Thus liftings corresponds to characters. By Lemma 2.4 the group of characters $\Gamma^\vee := \mathrm{Hom}(\Gamma, \mathbb{C}^*)$ is isomorphic to $\mathbb{Z}/12\mathbb{Z}$. ■.

References

[HB] Hannay J.H. and Berry M.V., Quantization of linear maps on the torus - Fresnel diffraction by a periodic grating. *Physica D1* (1980), 267-291.

[CM] Coxeter H. S. M. and Moser, W. O. J., Generators and relations for discrete groups. *Ergebnisse der Mathematik und ihrer Grenzgebiete, 14. Springer-Verlag, Berlin-New York* (1980).

[D] Degli Esposti M., Quantization of the orientation preserving automorphisms of the torus. *Ann. Inst. Henri Poincare 58* (1993), 323-341.

[D4] Deligne P., Note on quantization. *Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), Amer. Math. Soc. Providence, RI* (1999) 367–375.

[KR1] Kurlberg P. and Rudnick Z., Hecke theory and equidistribution for the quantization of linear maps of the torus. *Duke Math. Jour. 103* (2000), 47-78. Series, 33. Princeton University Press, Princeton, N.J. (1980).

[Me] F. Mezzadri, On the multiplicativity of quantum cat maps. *Nonlinearity 15* (2002), 905-922.

[MR] McConnell J.C. and Robson J.C., Non-commutative Noetherian Rings. *Graduate studies in mathematics, 30. Amer. Math. Soc. Providence, RI* (2001), 343-344.

[Ri] Rieffel M.A., Non-commutative tori—a case study of non-commutative differentiable manifolds, *Contemporary Math. 105* (1990), 191-211.